$$f(x) = \arctan x + C$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)} = \arctan x + C$$

$$x = 0 \Rightarrow 0 = \arctan 0 + C$$

$$0 = 0 + C$$

$$\hookrightarrow C = 0$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)} = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots = \arctan x \quad , |x| < 1$$

Last Time

power series: $\sum_{n=0}^{\infty} a_n x^n = f(x) \xrightarrow{\text{derivative}} \sum_{n=1}^{\infty} a_n \cdot n x^{n-1} = f'(x)$

$\xrightarrow{\text{integral}} \sum_{n=0}^{\infty} \frac{a_n}{n+1} \cdot x^{n+1} + C = \int f(x)\,dx$

ex: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

Not equal for all x

Just equal for $-1 < x < 1$

ex: $f(x) = \frac{5}{3-x}$ Find the power series about $c=0$ of $f(x)$, find the interval of convergence.

$$f(x) = \frac{5}{3-x} = \sum_{n=0}^{\infty} a_n \cdot x^n$$

10.8 TAYLOR AND MACLAURIN SERIES

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots$$

$a_n = ?$

$a_0 = f(c)$

$f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \dots$

$a_1 = f'(c)$

$f''(x) = 2a_2 + 6a_3(x-c) + \dots$

$2a_2 = f''(c)$

$$a_2 = \frac{f''(c)}{2} \qquad a_3 = \frac{f'''(c)}{6}$$

$$a_n = \frac{f^{(n)}(c)}{n!}$$

ex: $\underbrace{\frac{1}{1-x}}_{f(x)} = \sum_{n=0}^{\infty} x^n \quad , \quad |x| < 1$

$a_n = 1$

$f^{(n)}(0) = n!$ let's check!

$f(0) = 1$

$f'(x) = \frac{1}{(1-x)^2} \qquad f'(0) = 1$

$f''(x) = \frac{2}{(1-x)^3} \qquad f''(0) = 2$

$f''(x) = \frac{6}{(1-x)^4} \qquad f'''(0) = 6$

Definition:

* $f \to$ infinitely times differentiable at c.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

is called the Taylor series of f at $x=c$.

* If $c=0 \Rightarrow$ the series is called Maclaurin series of c.

ex: Maclaurin series of $f(x) = \frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$

$$\Rightarrow f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1$$

ex: $f(x) = e^x$ $x=0$ Taylor series?

$$a_n = \frac{f^{(n)}(0)}{n!} \quad \text{(nth derivative)}$$

(0th derivative)

$$f^{(0)}(0) = f(0) = e^0 = 1 \Rightarrow a_0 = \frac{1}{0!} = 1$$

$$f'(x) = e^x, \quad f'(0) = 1$$

$$a_1 = f'(0) = 1$$

$$f''(x) = e^x \quad f''(0) = 1$$

$$a_2 = \frac{f''(0)}{2!} = \frac{1}{2}$$

$$f^{(n)}(x) = e^x \quad \Big\} \quad f^{(n)}(0) = 1 \qquad \boxed{a_n = \frac{1}{n!}}$$

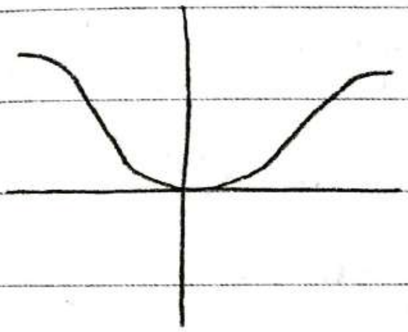
Maclaurin series of e^x is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

** Is it always true that a function is equal to its Taylor series?

NOT ALWAYS

ex: $f(x) = \begin{cases} 0 & , x = 0 \\ e^{-1/x^2} & x \neq 0 \end{cases}$



$0 = f(0) = f'(0) = f''(0)$

Its Taylor series is

$0 + 0x + 0x^2 + 0x^3 + \dots$

Function equals its Taylor series only at $x = 0$

ex: Find the Taylor series of $f(x) = \sin x$ at $x = 0$

$f(0) = \sin 0 = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$f''''(x) = \sin x \qquad f''''(0) = 0$

$a_n = f^{(n)}(0)$

$0 + 1 \cdot x + 0 - \frac{1}{3!} \cdot x^3 + 0 + \frac{x^5}{5!} + 0 + \frac{x^7}{7!} + \dots$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} = 1 \cdot \frac{x}{1!}$$

* what is the interval of convergence of this series?

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{2(n+1)+1} / (2(n+1)+1)!}{x^{2n+1} / (2n+1)!} \right|$$

$$= \lim_{n\to\infty} |x|^2 \cdot \frac{(2n+1)!}{(2n+3)!}$$

$$= |x|^2 \cdot \lim_{n\to\infty} \frac{1}{(2n+3)(2n+2)} = 0$$

Ex: Find the Maclaurin series of $\sin 3x$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n+1}}{(2n+1)!} \qquad x \in (-\infty, \infty)$$

$$\sin 3x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (3x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^{2n+1}}{(2n+1)!} \cdot x^{2n+1}$$

ex: Find the Taylor series of e^{2x} at $x=3$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in (-\infty, \infty)$$

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n \cdot x^n}{n!}$$

$$e^{2(x-3)} = \sum_{n=0}^{\infty} \frac{2^n \cdot (x-3)^n}{n!}$$

$$e^{2x} = \sum_{n=0}^{\infty} \frac{e^6 \cdot 2^n}{n!} (x-3)^n$$

ex: Find the Maclaurin series of $\cos x$.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\Big\downarrow \frac{d}{dx}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{(2n)!} \quad , \quad x \in (-\infty, \infty)$$

ex: Find the Maclaurin series of

$$\frac{2+x}{1-x}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad , \quad |x| < 1$$

$$\frac{2+x}{1-x} = (2+x)\sum_{n=0}^{\infty} x^n = (2+x)(1+x+x^2+x^3+\cdots)$$

$$= (2+2x+2x^2+2x^3+\cdots)+(1+x+x^2+x^3+\cdots)$$

$$= 2+3x+3x^2+3x^3+3x^4+\cdots$$

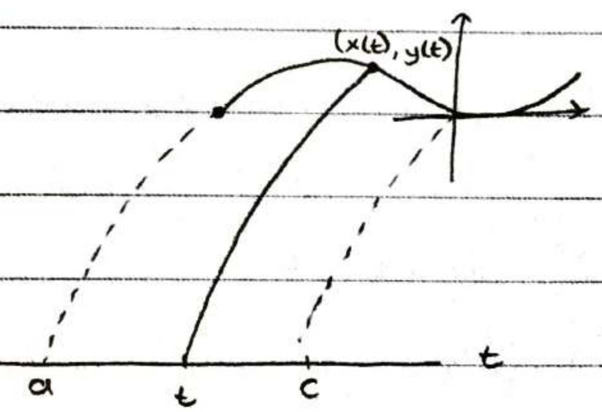
$(+1-1)$

$$= -1+3(1+x+x^2+\cdots)$$

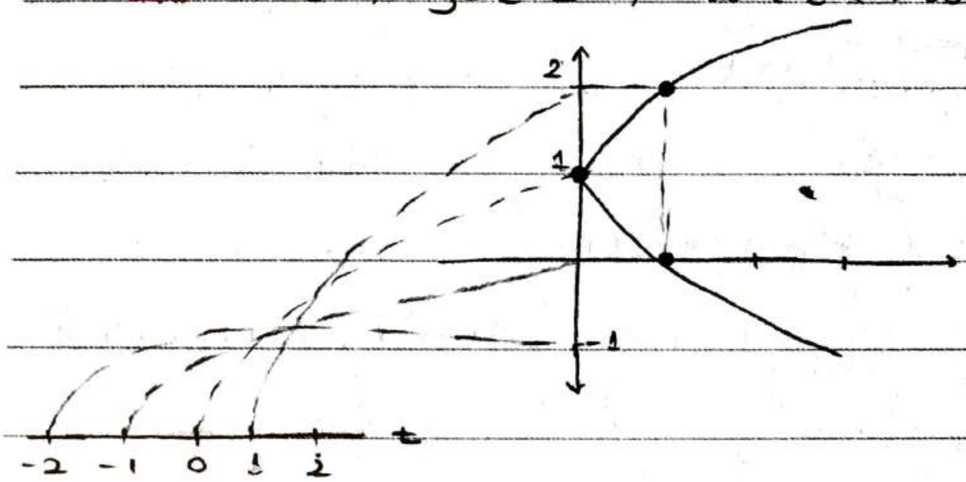
$$= -1+\sum_{n=0}^{\infty} x^n$$

CHAPTER 11 - PARAMETRIC EQUATIONS AND POLAR COORDINATES

11.1 PARAMETRIZATION OF CURVES

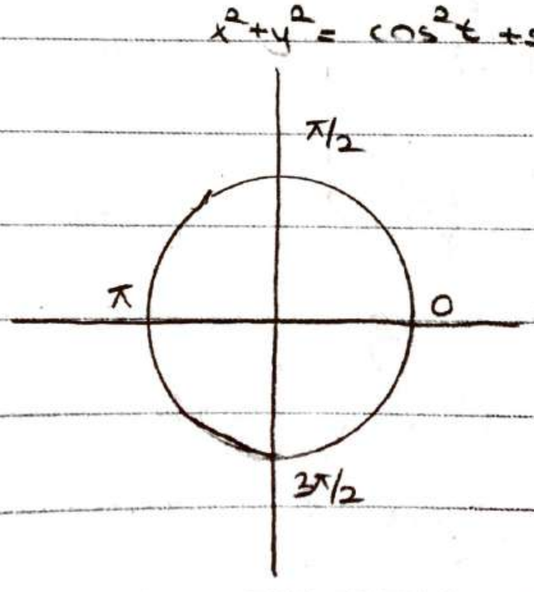


ex: $x = t^2$, $y = t+1$, $-\infty < t < +\infty$

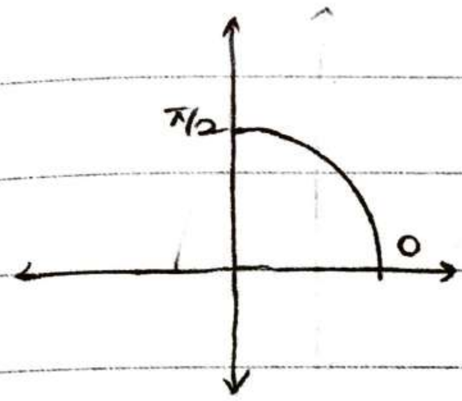


ex $x = \cos t$, $y = \sin t$ $\quad 0 \leq t \leq 2\pi$

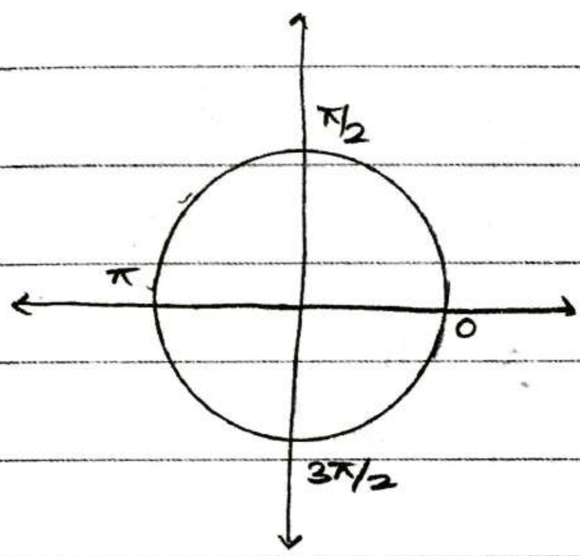
$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$



ex: $x = \cos t$, $y = \sin t$ $\quad 0 \leq t \leq \pi/2$



ex: $x = \cos 2t$ $\quad y = \sin 2t$ $\quad 0 \leq t \leq \pi$



ex: $a > 0$

$x = a\cos t$

$y = a\sin t$

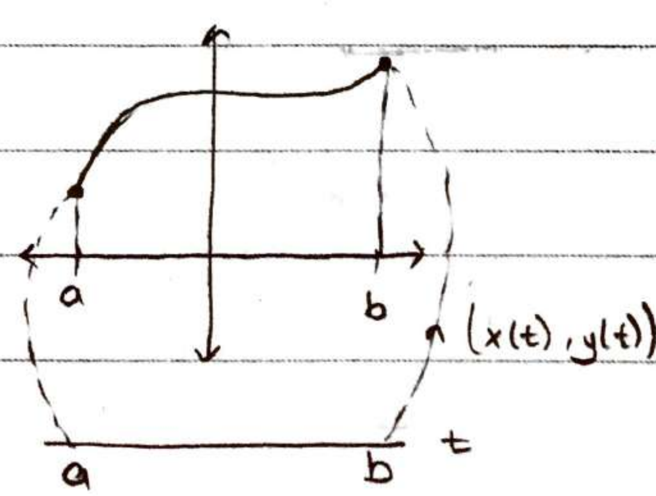
$0 \leq t \leq 2\pi$

$x^2 + y^2 = a^2$

ex: The graph of function $y = f(x)$ can be parametrized.

$x = t$

$y = f(t)$



ex: $y=x^2 \Leftrightarrow$ $x=t$

$y=t^2$

$-\infty < t < +\infty$

ex: $x = t + \frac{1}{t}$

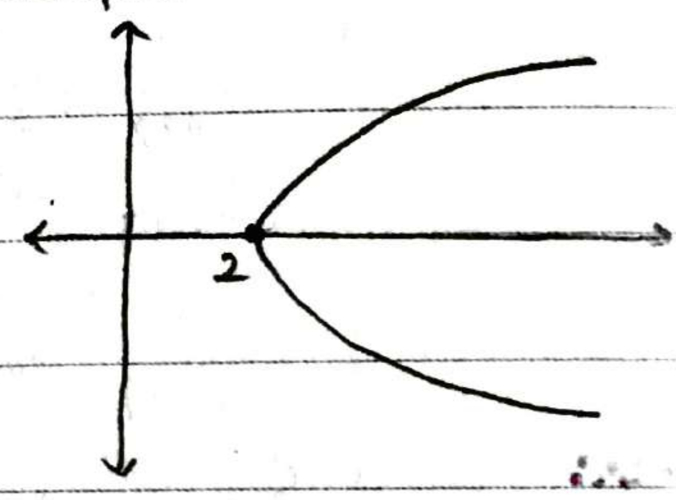
$y = t - \frac{1}{t}$ Find y in term of x $(t>0)$

$x+y = 2t$

$x-y = \frac{2}{t}$

$(x+y)(x-y) = 2t \cdot \frac{2}{t} = 4$

$x^2 - y^2 = 4$



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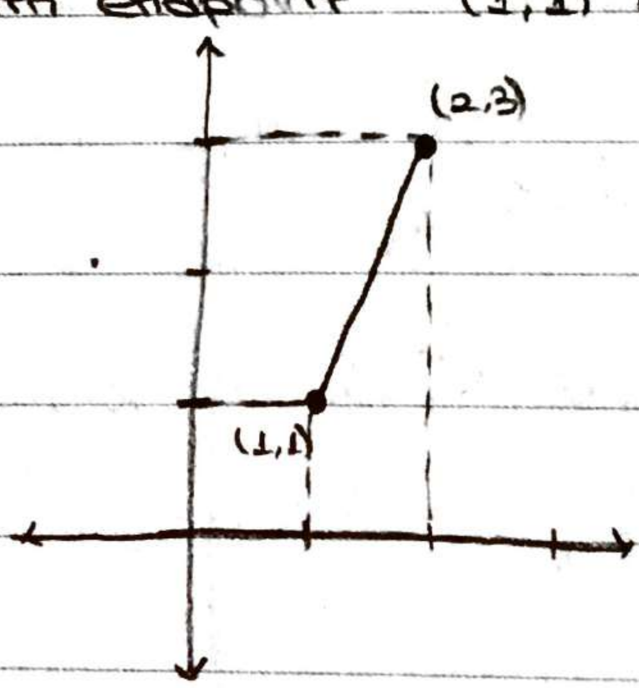
ex: Find the parametrization for the line segment with endpoint $(1,1)$ and $(2,3)$



$m = \frac{3-1}{2-1} = 2$

$y-1 = 2(x-1)$, $1 \leq x \leq 2$

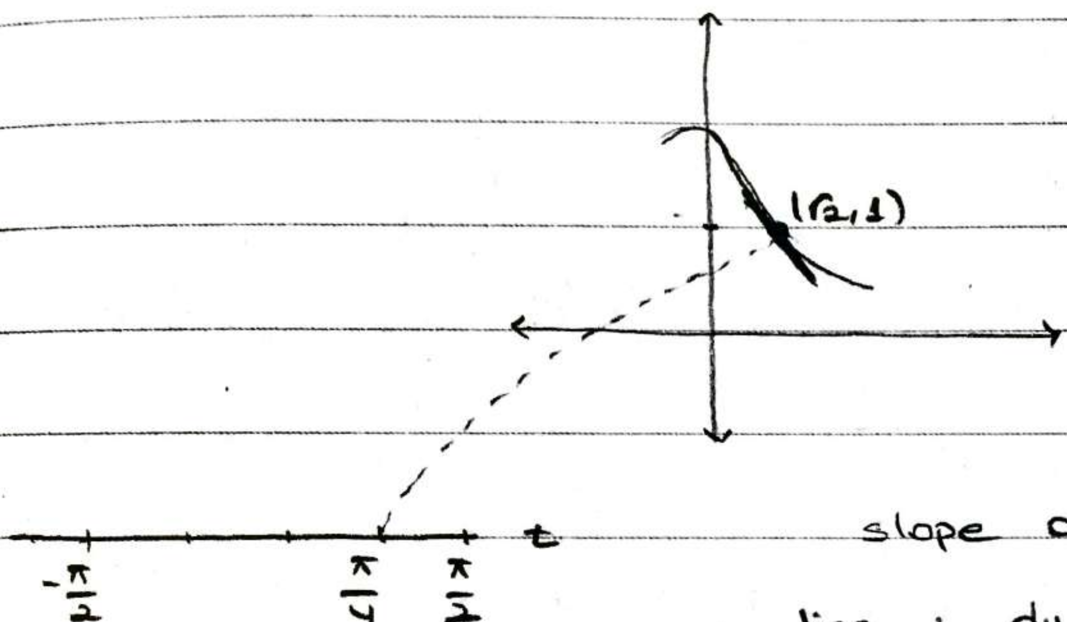
✓ $\begin{cases} x = t \\ y = 2(t-1)+1 \\ 1 \leq t \leq 2 \end{cases}$ ✓ $\begin{cases} x = \frac{t-1}{2} + 1 \\ y = t \\ 1 \leq t \leq 3 \end{cases}$

11.2 CALCULUS WITH PARAMETRIC CURVES

ex: Find the tangent line to the curve

$x = \sec t \quad , \quad y = \tan t \quad , \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$

at the point $(\sqrt{2}, 1)$ where $t = \frac{\pi}{4}$



slope of the tangent line : $\left.\frac{dy}{dx}\right|_{(\sqrt{2},1)}$

$$\frac{dy}{dt} = \boxed{\frac{dy}{dx}} \cdot \frac{dx}{dt}$$

$$\sec^2 t = \frac{dy}{dx} \cdot \sec t \cdot \tan t$$

$$\frac{dy}{dx} = \frac{\sec t}{\tan t}$$

$$\left.\frac{\sec t}{\tan t}\right|_{t=\frac{\pi}{4}} = \frac{1/\cos \pi/4}{\sin \pi/4 / \cos \pi/4} = \sqrt{2} \rightarrow \text{slope}$$

$$\boxed{y - 1 = \sqrt{2}\,(x - \sqrt{2})}$$

$x = x(t)$

$y = y(t)$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$$

$$\frac{dy}{dx} = y'$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt}}$$

ex: Find $\frac{d^2y}{dx^2}$ as a function of t if $x = t - t^2$ and $y = t - t^3$

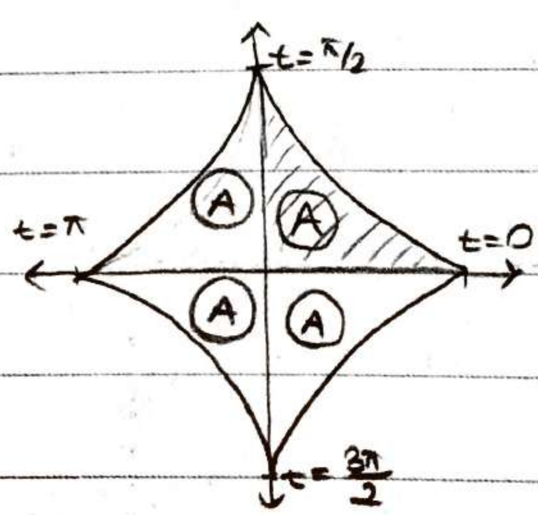
$$\frac{dy}{dx} = y' = \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{1-2t}$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\frac{(-6t)(1-2t) - (1-3t^2)(-2)}{}}{1-2t}$$

$$= \frac{2-6t+6t^2}{(1-2t)^3}$$

ex: Find the area enclosed by the astroid

$x = \cos^3 t$, $y = \sin^3 t$, $0 \leq t \leq 2\pi$



$$\text{Area} = \int_{x=0}^{1} y(x)\, dx$$

$dx = 3\cos^2 t\,(-\sin t)\,dt$

$$\int_{t=\pi/2}^{0} \sin^3 t \cdot 3\cos^2 t\,(-\sin t)\,dt$$

$$= \int_{0}^{\pi/2} 3\sin^4 t \cdot \cos^2 t\,dt$$

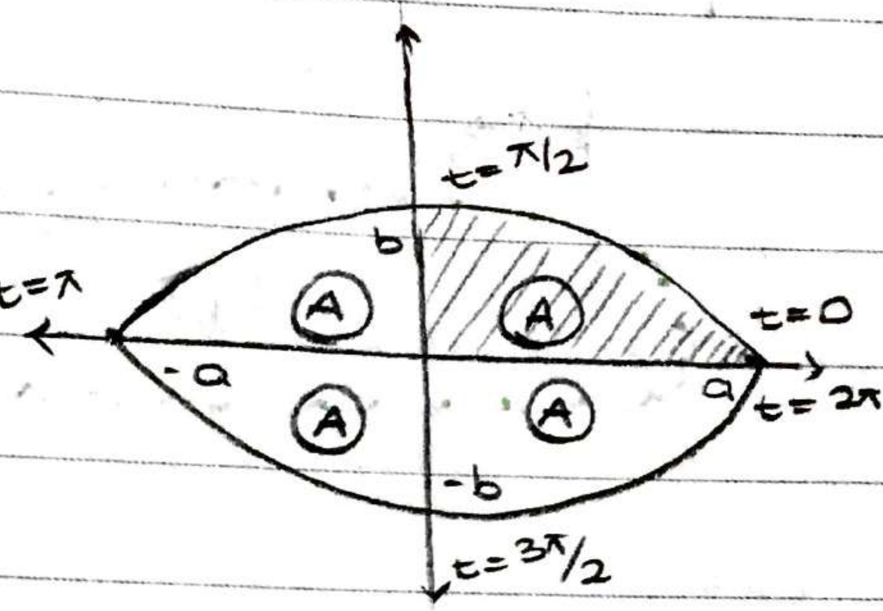
$$= \frac{3\pi}{32} \Rightarrow \text{total Area} = \frac{3\pi}{8}$$

ex: Find the area enclosed by the ellipse

$x = a\cos t$

$y = b\sin t$

$0 \leq t \leq 2\pi$



$$\text{Area} = \int_{x=0}^{a} y \cdot dx = \int_{t=\pi/2}^{0} \underbrace{(b\sin t)}_{y} \cdot a \cdot (-\sin t)\, dt$$

$$= a \cdot b \int_{0}^{\pi/2} \sin^2 t\, dt$$

$$= a \cdot b \int_{0}^{\pi/2} \left(\frac{1-\cos 2t}{2}\right) dt$$

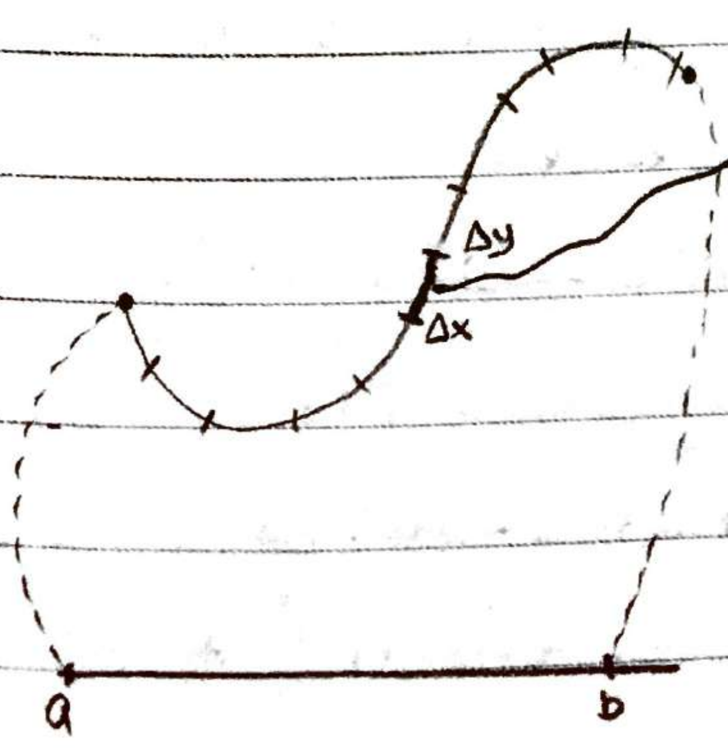
$$= a \cdot b \left(\frac{t}{2} - \frac{\sin 2t}{4}\right) \Bigg|_{0}^{\pi/2}$$

$$= a \cdot b \cdot \left(\left(\frac{\pi}{4} - \frac{\sin \pi}{4}\right) - (0-0)\right)$$

$$= \frac{ab\pi}{4}$$

$$\text{Total Area} = 4 \cdot \frac{ab\pi}{4} = a \cdot b\pi$$

!!



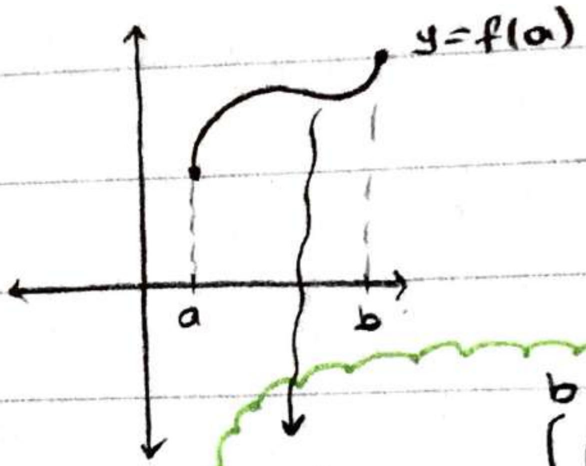
$$\text{length} \approx \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\approx \sqrt{\left(\frac{dx}{dt}\right)^2 (\Delta t)^2 + \left(\frac{dy}{dt}\right)^2 (\Delta t)^2}$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot \Delta t$$

$$\sum_{i=1}^{n} \sqrt{\left(\frac{dx}{dt}\right)^2 (t_i) + \left(\frac{dy}{dt}\right)^2 (t_i)} \cdot \Delta t$$

$$\xrightarrow{n \to \infty} \boxed{\int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$ length of the curve



$$\text{arc length} = \int_a^b \sqrt{1+f'(x)^2}\,dx$$

ex: $x = t$, $y = f(t)$, $a \leq t \leq b$

$$\text{arc length} = \int_{t=a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_{x=a}^{b} \sqrt{1+f'(x)^2}\,dt$$

Area of Surface of Revolution

1) Revolution about the x-axis:

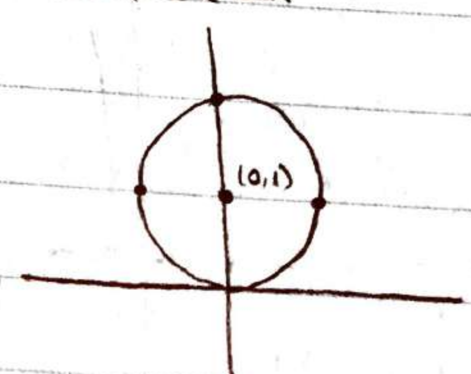
$$\text{Area} = \int_{t=a}^{b} 2\pi \cdot y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

2) Revolution about the y-axis:

$$\text{Area} = \int_{t=a}^{b} 2\pi\, x(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

ex: $x = \cos t$, $y = 1+\sin t$, $0 \leq t \leq 2\pi$

Find the area of the surface swept out by revolving the curve about the x-axis



$$x^2 + (y-1)^2 = \cos^2 t + \sin^2 t = 1$$

$$\text{Area} = \int_0^{2\pi} 2\pi(1+\sin t)\sqrt{(1-\sin^2 t) + (\cos t)^2}\,dt$$

$$= \int_0^{2\pi} 2\pi(1+\sin t)\,dt$$

$$= 2\pi \cdot (t - \cos t)\Big|_0^{2\pi} \Rightarrow 4\pi^2$$

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ex: Find the length of the curve

$x = t^3 \qquad y = \frac{3t^2}{2} \quad , \quad 0 \le t \le \sqrt{3}$

$$\text{length} = \int_0^{\sqrt{3}} \sqrt{\underbrace{(3t^2)^2}_{\left(\frac{dx}{dt}\right)^2} + \underbrace{(3t)^2}_{\left(\frac{dy}{dt}\right)^2}}\, dt$$

$$= 3 \int_0^{\sqrt{3}} \sqrt{t^4 + t^2}\, dt$$

$$= 3 \cdot \int_0^{\sqrt{3}} t\sqrt{t^2+1}\, dt$$

$t^2 + 1 = u \qquad 2t\,dt = du$

$$= 3 \int_1^4 \sqrt{u}\, \frac{du}{2} = \frac{3}{2} \cdot \left.\frac{u^{3/2}}{3/2}\right|_1^4 = 4^{3/2} - 1 = 7$$